The Accuracy of Linear Piston Theory When Applied to Cylindrical Shells

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PISTON theory was introduced into aeroelasticity in the linearized form by Ashley and Zartarian[1] as a handy tool in 1956. This theory furnishes an approximation for the aerodynamic pressure acting on a slightly deformed flat plate in a supersonic airstream. The linearized piston theory is used widely in the investigation of the flutter of flat panels. Because of the lack of appropriate approximations for the aerodynamic pressure acting on a vibrating cylindrical shell, linear piston theory is used also in investigations of the flutter of cylindrical shells.[3, 4]

There are doubts about the accuracy of using the linear piston theory for cylindrical shells. This opinion was strengthened by recent studies at California Institute of Technology.[4, 5, 8] In the following, a short survey of an investigation of the accuracy of the linear piston theory when applied to cylindrical shells is given. For further details, the reader is referred to Ref. 6.

An infinitely long circular cylindrical shell of radius R, which is exposed externally to a uniform airstream parallel to the cylinder axis, is considered. The Mach number, the density, and the velocity of sound of the undisturbed airstream are denoted by M, ρ_0, and a_0, respectively. Let a cylinder coordinate system x, r, θ be chosen, where the positive direction of the x axis coincides with the positive direction of the airstream (i.e., a negative Mach number means that the airstream is moving in the negative x direction). The shell is assumed to be slightly deformed by a harmonically oscillating standing wave of the form

$$w = w_0 \cos(n\theta)s(x)e^{i\omega t} \qquad \omega \geqslant 0 \qquad (1a)$$

with

$$s(x) = \sin(\nu x) \text{ or } s(x) = \cos(\nu x) \qquad \nu \gtrless 0 \qquad (1b)$$

The radial displacement of the shell, w, is measured positive in the outside direction. By proper superposition of the formulas, set forth by Leonhard and Hedgepeth in Ref. 7, one obtains an exact expression for the aerodynamic pressure Δp, which is originated by the shell vibration (1). Two of the main parameters of these expressions are

$$M_1 = M - (\omega/\nu a_0) \qquad M_2 = M + (\omega/\nu a_0) \qquad (2)$$

These exact expressions involve fractions of cylinder functions and their first derivatives. Applying the well-known asymptotic expansions for cylinder functions (see Ref. 2), one obtains for these fractions the following asymptotic expansions for $\xi \to \infty$ (ξ is restricted to real, positive values):

$$\frac{H_n^{(1)}(\xi)}{H_n^{(1)\prime}(\xi)} = -i - \frac{1}{2}\xi^{-1} - i\left(\frac{n^2}{2} - \frac{3}{8}\right)\xi^{-2} - \left(n^2 - \frac{3}{8}\right)\xi^{-3} + O(\xi^{-4}) \qquad (3a)$$

$$\frac{H_n^{(2)}(\xi)}{H_n^{(2)\prime}(\xi)} = \overline{\left(\frac{H_n^{(1)}(\xi)}{H_n^{(1)\prime}(\xi)}\right)} \qquad (3b)$$

$$\frac{K_n(\xi)}{K_n'(\xi)} = -1 + \frac{1}{2}\xi^{-1} + \left(\frac{n^2}{2} - \frac{3}{8}\right)\xi^{-2} - \left(n^2 - \frac{3}{8}\right)\xi^{-3} + O(\xi^{-4}) \quad \text{for } \xi \to \infty \qquad (3c)$$

Here and in the following, O denotes the Landau symbol. [The barred symbol in (3b) denotes the conjugate complex term.]

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The asymptotic expansions (3) lead to asymptotic expansions for the aerodynamic pressure Δp. These expansions allow an investigation of the accuracy of the linear piston theory approximation

$$\Delta p^{**} = a_0\rho_0\{a_0M(\partial w/\partial x) + (\partial w/\partial t)\} \qquad (4)$$

when applied to cylindrical shells. Furthermore, improved approximations can be obtained from the asymptotic expansions of Δp.

For

$$|M_1| > 1 \qquad |M_2| > 1 \qquad (5)$$

one obtains finally, separating the linear piston theory expression (4),

$$\Delta p = a_0\rho_0\left\{a_0M\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t}\right\} + \sum_{m=0}^{3}\frac{1}{R^m}\left(F_m w + F_m^*\frac{\partial w}{\partial x}\right) + w\left[O\left(\frac{1}{\nu^3R^4}\frac{1}{M_1^3[1-(1/M_1^2)]^{5/2}}\right) + O\left(\frac{1}{\nu^3R^4}\frac{1}{M_2^3[1-(1/M_2^2)]^{5/2}}\right)\right] + \frac{\partial w}{\partial x}\left[O\left(\frac{1}{\nu^4R^4}\frac{1}{M_1^3[1-(1/M_1^2)]^{5/2}}\right) + O\left(\frac{1}{\nu^4R^4}\frac{1}{M_2^3[1-(1/M_2^2)]^{5/2}}\right)\right] \qquad (6a)\dagger$$

with

$$F_0 = \frac{1}{2}i\nu a_0^2\rho_0\left(-\frac{M_1}{[1-(1/M_1^2)]^{1/2}} + \frac{M_2}{[1-(1/M_2^2)]^{1/2}} - \frac{2\omega}{\nu a_0}\right) \qquad (6b)$$

$$F_0^* = \frac{1}{2}a_0^2\rho_0\left(\frac{M_1}{[1-(1/M_1^2)]^{1/2}} + \frac{M_2}{[1-(1/M_2^2)]^{1/2}} - 2M\right) \qquad (6c)$$

$$F_1 = -\frac{1}{4}a_0^2\rho_0\left(\frac{1}{1-(1/M_1^2)} + \frac{1}{1-(1/M_2^2)}\right) \qquad (6d)$$

$$F_1^* = \frac{i}{4\nu}a_0^2\rho_0\left(-\frac{1}{1-(1/M_1^2)} + \frac{1}{1-(1/M_2^2)}\right) \qquad (6e)$$

$$F_2 = \frac{i}{4\nu}\left(n^2 - \frac{3}{4}\right)a_0^2\rho_0\left(-\frac{1}{M_1[1-(1/M_1^2)]^{3/2}} + \frac{1}{M_2[1-(1/M_2^2)]^{3/2}}\right) \qquad (6f)$$

$$F_2^* = \frac{1}{4\nu^2}\left(n^2 - \frac{3}{4}\right)a_0^2\rho_0\left(\frac{1}{M_1[1-(1/M_1^2)]^{3/2}} + \frac{1}{M_2[1-(1/M_2^2)]^{3/2}}\right) \qquad (6g)$$

$$F_3 = -\frac{1}{2\nu^2}\left(n^2 - \frac{3}{8}\right)a_0^2\rho_0\left(\frac{1}{M_1^2[1-(1/M_1^2)]^2} + \frac{1}{M_2^2[1-(1/M_2^2)]^2}\right) \qquad (6h)$$

$$F_3^* = \frac{i}{2\nu^3}\left(n^2 - \frac{3}{8}\right)a_0^2\rho_0\left(-\frac{1}{M_1^2[1-(1/M_1^2)]^2} + \frac{1}{M_2^2[1-(1/M_2^2)]^2}\right) \qquad (6i)$$

† Here and in the following, the positive square root always is chosen.

The terms $(1/R^m)F_m w$ and $(1/R^m)\, F_m^*(\partial w/\partial x)$, $m = 0, \ldots, 3$, in (6a) are called "correction terms"; the following terms are referred to as "remainder." It is obvious that the remainder tends to zero as soon as one of the parameters M, R, ω, and $|\nu|$ tends to infinity.

An inspection of Eqs. (6) leads to the following conclusions:

$$\Delta p - \left\{\Delta p^{**} - \frac{a_0^2\rho_0}{2R}\, w\right\} \to 0 \quad \text{for } |M| \to \infty \tag{7}$$

$$\Delta p - \left[\Delta p^{**} - \frac{1}{2}\, a_0\rho_0 \left\{\frac{1}{M_1M_2} + \left|\binom{-\frac{1}{2}}{2}\right| \frac{(\nu a_0/\omega)(M_2^3 - M_1^3)}{M_1^3M_2^3} + \left|\binom{-\frac{1}{2}}{3}\right| \frac{(\nu a_0/\omega)(M_2^5 - M_1^5)}{M_1^5M_2^5} + \ldots\right\} \frac{\partial w}{\partial t} + \frac{1}{2}\, a_0^2\rho_0 M \left\{\frac{1}{M_1M_2} + \left|\binom{-\frac{1}{2}}{2}\right| \frac{M_2^3 + M_1^3}{MM_1^3M_2^3} + \left|\binom{-\frac{1}{2}}{3}\right| \frac{M_2^5 + M_1^5}{MM_1^5M_2^5} + \ldots\right\} \frac{\partial w}{\partial x}\right] \to 0 \quad \text{for } R \to \infty \tag{8}$$

The latter result is obtained after F_0 and F_0^* have been expanded in power series of $1/M_1$ and $1/M_2$ and after some regrouping. Obviously, the terms $M_2^\mu - M_1^\mu$, $\mu = 3, 5, \ldots$, contain the factor $\omega/\nu a_0$. The terms $M_2^\mu + M_1^\mu$, $\mu = 3,5, \ldots$, contain the factor M. The relations (7) and (8) lead to

$$\Delta p - \Delta p^{**} \to 0 \quad \text{for } |M| \to \infty;\ R \to \infty \tag{9}$$

Furthermore, one obtains

$$\Delta p - \{\Delta p^{**} - (a_0^2\rho_0/2R)w\} \to 0 \quad \text{for } \omega \to \infty \tag{10}$$

Hence

$$\Delta p - \Delta p^{**} \to 0 \quad \text{for } R \to \infty;\ \omega \to \infty \tag{11}$$

In order to investigate the limiting process $|\nu| \to \infty$, M is restricted by $|M| > 1$. Thanks to this inequality, the inequalities (5) are satisfied for sufficiently large values of $|\nu|$. Then one learns that

$$\Delta p - \left[\Delta p^{**} + a_0\rho_0 \left(\frac{|M|}{(M^2-1)^{1/2}}\, \frac{M^2-2}{M^2-1} - 1\right) \frac{\partial w}{\partial t} + a_0^2\rho_0 \left(\frac{M|M|}{(M^2-1)^{1/2}} - M\right) \frac{\partial w}{\partial x} - \frac{a_0^2\rho_0}{2R}\, \frac{M^2}{M^2-1}\, w\right] \to 0 \quad \text{for } |\nu| \to \infty \tag{12}$$

From (12) one arrives at

$$\Delta p - \{\Delta p^{**} - (a_0^2\rho_0/2R)w\} \to 0 \quad \text{for } |M| \to \infty,\ |\nu| \to \infty \tag{13}$$

$$\Delta p - \Delta p^{**} \to 0 \quad \text{for } |M| \to \infty,\ |\nu| \to \infty,\ R \to \infty \tag{14}$$

These investigations demonstrate that, for the case $|M_1| > 1$; $|M_2| > 1$, the linear piston theory expression Δp^{**} can be considered as a first-order approximation of the aerodynamic pressure Δp. Nevertheless, the replacement of Δp^{**} by the approximation

$$\Delta p^{***} = \Delta p^{**} - \frac{a_0^2\rho_0}{2R}\, w = a_0\rho_0 \left\{a_0 M \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} - \frac{a_0}{2R}\, w\right\} \tag{15}$$

is suggested as a first-order step of improvement for the application to cylindrical shells. Improved approximations can be obtained from Eqs. (6). For large values of n, especially, the correction terms of the order $(1/R)^2$ and $(1/R)^3$ in Eqs. (6) can become quite significant. For flutter investigations, it often is necessary to consider values of n up to the order of 20.

Further investigations along this line (see Ref. 6) disclose that, in cases 1) $|M_1| < 1$, $|M_2| < 1$; 2) $|M_1| < 1$, $|M_2| > 1$; and 3) $|M_1| > 1$, $|M_2| < 1$, the linear piston theory expression Δp^{**} no longer can be considered as a first-order approximation for the aerodynamic pressure Δp.

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Transient Temperature of a Porous-Cooled Wall

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Nomenclature

C = specific heat, Btu/lb-°F
G = mass flow rate, lb/hr-ft²
k = thermal conductivity, Btu/hr-ft-°F
L = thickness, ft
q = heat flux, Btu/hr-ft²
t = temperature, °F
x = normal coordinate through wall, ft
θ = time, hr
ρ = density, lb/ft³

Subscripts

c = coolant
w = porous solid

THIS note presents parametric curves, based on an exact solution, for the prediction of the transient temperature response of a one-dimensional transpiration-cooled porous wall that is exposed instantaneously to a constant heat flux on the coolant exit surface. For short-time applications,

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